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A GENERALIZED CAUCHY PROBLEM FOR THE LINEAR DIFFERENTIAL EQUATIONS OF COUPLED PHYSICAL - MECHANICAL FIELDS*

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A generalized Cauchy problem for a partial differential equation with constant coefficients, which is encountered in the study of physical processes in continuous media with widened physical - mathematical fields (see /1/) (generalized coupled thermoeleasticity /2/, coupled thermoeleasticity, porous media saturated with a viscous fluid /5/, mass and heat transfer /6/, linearized magnetoelasticity /7/, etc.) is considered. The characteristic properties of the solution of the problem, under certain constraints imposed on an equation by the stability condition, are studied. The presence of waves of higher and lower order is characteristic for the solution; in the course of time the lower-order waves are maintained and take a characteristic form. In the general case, the solution is represented in the form of integrals over the segments which link the singular points of Fourier - Laplace transforms with respect to time of the solution under consideration. The methods proposed enable an exact investigation to be made of the processes described by the equation for any time constants, and they also enable one to isolate the singularities at the fronts of propagating perturbations. As an application, the dynamic processes taking place in a thermoelastic subsapce (2) as a result of applying a mechanical and a thermal input at the boundary is studied. It is shown that in the case of unit perturbation of the boundary, the stress and temperature waves in the course of time assume a bell-shaped form and propagate with adiabatic velocity. A numerical analysis of the process which occurs due to sudden application of the force and of the thermal shock at the boundary is given.

1. Consider a differential equation with constant coefficients of the form

$$M(\partial_\xi, \partial_\tau)\Phi(\xi, \tau) = \Psi(\xi, \tau), \quad \xi \in R^1, \quad \tau \in R^1 \tag{1.1}$$
$$M(\partial_\xi, \partial_\tau) = \partial_\xi^4 + (a_{21}\partial_\tau + a_{22}\partial_\tau^2)\partial_\xi^2 + a_{03}\partial_\tau^3 + a_{04}\partial_\tau^4$$
$$\partial_\xi = \partial/\partial\xi, \quad \partial_\tau = \partial/\partial\tau$$

where R^n is an n-dimensional real space. This equation is the most common for all the problems mentioned above.

The generalized Cauchy problem for Eq.(1.1) with the source $\Psi(\xi, \tau)$, $e^{-\omega_2\tau}\Psi(\xi, \tau) \in S'$, $\Psi(\xi, \tau) = 0$ when $\tau < 0$ is defined (see /8/) as the problem of finding the generalized function $\Phi(\xi, \tau)$, $e^{-\omega_2\tau}\Phi(\xi, \tau) \in S'$ which satisfies (1.1), and when $\tau < 0$ vanishes for a certain $\omega_2 > 0$. This is equivalent to finding a solution of (1.1) which satisfies the causality principle widely used in physics /9/. Here S' is the space of generalized slowly increasing functions, i.e. the space of linear continuous functionals in the space S of rapidly decreasing basic functions.

Let us find the solution $\Phi_l(\xi, \tau)$ $(l = 0, \ldots, 4)$ of the generalized Cauchy problem (1.1) with sources of the form

$$\Psi(\xi, \tau) = \delta^{(l)}(\xi) f(\tau), \quad l = 0, \ldots, 4 \tag{1.2}$$

where $f(\tau) \in D_+'(a_0)$ ($f(\tau) = 0$ for $\tau < 0$, $e^{-\omega_2\tau} f(\tau) \in S'$ for all $\omega_2 > a_0$), and $\delta(\xi)$ is the Dirac function. If $f(\tau) = \delta(\tau)$ then $\Phi_0(\xi, \tau)$ is the delayed Green function.

We shall construct the solution of (1.1) with the source (1.2) by using Fourier transformion with respect to the coordinate ξ, which we define by the relation

$$(F[\Phi], \varphi) = (\Phi, F[\varphi]), \quad \Phi \in S', \quad \varphi \in S$$

(see /8/), where

$$F[\varphi](k) = \int_{-\infty}^{+\infty} \varphi(\xi) e^{i\xi k} d\xi$$

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is the Fourier transform of the basic functions $\varphi \in S$, and the Fourier-Laplace transform with respect to time, which we determine from the formula

$$L[f](\omega) = F[f(\tau) e^{-\omega_2\tau}](\omega_1), \quad \omega_2 > a_0; \quad \omega = \omega_1 + i\omega_2,$$
$$f(\tau) \in D_+'(a_0)$$

Applying these transforms we obtain

$$L[F[\Phi_l]] = (-ik)^l L[f]/D(\omega, k), \quad D(\omega, k) \equiv \tag{1.3}$$
$$M(-ik, -i\omega) = (-ik)^4 + (-ik)^2 [a_{21}(-i\omega) + a_{22}(-i\omega)^2] + a_{03}(-i\omega)^3 + a_{04}(-i\omega)^4$$

The dispersion equation $D(\omega, k) = 0$, being a function of two complex variables ω and k, for $a_{04} \neq 0$ determines four zeros $\omega_j(k)$ $(j = 1, \ldots, 4)$ as functions of k, of four zeros $k_j(\omega)$ $(j = 1, \ldots, 4)$ as functions of ω. We write the latter in the form

$$k_{1,2}(\omega) = (\omega/2)^{1/2} [-a_{22}\omega - ia_{21} \pm \Omega(\omega)]^{1/2} \tag{1.4}$$
$$k_{3,4}(\omega) = -k_{1,2}(\omega)$$
$$\Omega(\omega) = [(a_{22}\omega + ia_{21})^2 - 4\omega(a_{04}\omega + ia_{03})]^{1/2} =$$
$$p_1^{1/2}(\omega - \omega_+)^{1/2}(\omega - \omega_-)^{1/2}, \quad \omega_\pm = -i\omega_2^\circ \pm \omega_1^\circ$$
$$\omega_1^\circ = 2p_2^{1/2}/p_1, \quad \omega_2^\circ = p_3/p_1, \quad p_1 = a_{22}^2 - 4a_{04}$$
$$p_2 = a_{22}a_{21}a_{03} - a_{21}^2 a_{04} - a_{03}^2, \quad p_3 = a_{22}a_{21} - 2a_{03}$$

We shall use the branch cuts, which connect the branch points $\omega_\pm$, $\omega = 0$, $\omega = -ia_{03}/a_{04}$ of the functions $k_1(\omega)$ and $k_2(\omega)$, in accordance with /10/ where the analytic properties of this type of function are disucssed, and fix the function branches by the condition $\operatorname{Im} k_{1,2}(\omega) \geqslant \text{const} > 0$ as $\operatorname{Im}\omega \to +\infty$. By the Routh-Hurwitz criterion,

$$\operatorname{Im}\omega_j(k) < 0, \quad j = 1, \ldots, 4, \quad k \in R^1, \quad k \neq 0 \tag{1.5}$$

with the following constraints imposed on the coefficients of Eq.(1.1):

$$a_{21} < 0, \quad a_{22} < 0, \quad a_{03} > 0, \quad a_{04} > 0, \quad p_2 > 0 \tag{1.6}$$

Clearly, condition (1.5) is a condition of the absolute stability of the system described by Eq.(1.1), e.g. from the fact that $\Psi(\xi, \tau) \in D_+'(a_0)$ with respect to τ it follows that $\Phi(\xi, \tau) \in D_+'(a_0)$ with respect to τ, where a_0 is an arbitrary positive number.

Following /11/, the equation for which inequality (1.5) is satisfied is referred to as the Petrovskii correct equation, and under the condition

$$a_{04} > 0, \quad a_{22} < 0, \quad p_1 > 0 \tag{1.7}$$

Eq.(1.1) is hyperbolic. On writing it in the form

$$[\eta(c_-^2\partial_\xi^2 - \partial_\tau^2)(c_+^2\partial_\xi^2 - \partial_\tau^2) - \partial_\tau(c_0^2\partial_\xi^2 - \partial_\tau^2)]\,\Phi(\xi, \tau) = \Psi(\xi, \tau) \tag{1.8}$$
$$c_\pm = [2(-a_{22} \pm \sqrt{p_1})]^{1/2}, \quad c_0 = [-a_{21}a_{03}]^{1/2}, \quad \eta = a_{04}a_{03}$$

we see that the stability of its solution follows from the satisfaction of the conditions

$$c_- > c_0 > c_+ > 0, \quad \eta > 0 \tag{1.9}$$

(see /12/). It can be shown that these conditions are equivalent to (1.6) and (1.7).

In addition, it can be shown by analogy with /13/ that if conditions (1.6), (1.7) are satisfied, we have

$$\operatorname{Im} k_{1,2}(\omega) > 0 \quad \text{for} \quad \operatorname{Im}\omega > 0 \tag{1.10}$$

that is space attenuation of waves occurs (the outside signal does not increase in proportion to the distance from the boundary).

By performing an inverse Fourier transformation, taking condition (1.10) into account, the solution $\Phi_l(\xi, \tau) = \partial_\xi^l \Phi_0(\xi, \tau)$ $(l = 0, \ldots, 4)$ of (1.1) with sources (1.2) can be expressed in terms of the five functions $G_j(|\xi|, \tau, n)$ $(j = 1, 2, 3, 5, 6)$ in the form

$$\Phi_0 = -{}^1\!/_2 G_3(|\xi|, \tau, 2) * f(\tau), \quad \Phi_1 = {}^1\!/_2 G_1(|\xi|, \tau, 1) * f(\tau)\operatorname{sign}\xi \tag{1.11}$$
$$\Phi_2 = -{}^1\!/_2 G_5(|\xi|, \tau, 0) * f(\tau), \quad \Phi_3 = {}^1\!/_4 [-a_{22}G_1(|\xi|, \tau, -1) - a_{21}G_1(|\xi|, \tau, 0) - G_2(|\xi|, \tau, -1)] * f(\tau)\operatorname{sign}\xi$$
$$\Phi_4 = {}^1\!/_4 [a_{22}G_5(|\xi|, \tau, -2) + a_{21}G_5(|\xi|, \tau, -1) - G_6(|\xi|, \tau, -2)] * f(\tau) + \delta(\xi) f(\tau)$$

where the functions $G_j(|\xi|, \tau, n)$ are written using the inversion formula for Fourier-Laplace transforms, (see /8/),

$$G_j(|\xi|, \tau, n) = G_{j1}(|\xi|, \tau, n) + G_{j2}(|\xi|, \tau, n) \tag{1.12}$$

$$G_{jp}(|\xi|,\tau,n)=\frac{1}{2\pi}\left(\frac{d}{d\tau}-a_0\right)^{m+2}\int_{-\infty+i\omega_2}^{\infty+i\omega_2}\frac{G^L_{jp}(\xi,\omega,n)}{(-i\omega-ia_0)^{m+2}}\exp(-i\omega\tau)\,d\omega$$
$$G_{jp}{}^L(\xi,\omega,n)\equiv L[G_{jp}](\omega)=R_{jp}(\omega)\exp[ik_p(\omega)|\xi|]/(-i\omega)^{n+1}$$
$$\omega_2>\omega_2'>a_0,\quad m=m(\omega_2'),\quad p=1,2,\quad j=1,2,3,5,6$$
$$R_{1p}(\omega)=(-1)^{p+1}\omega/\Omega(\omega),\ R_{2p}(\omega)=1,\quad R_{3p}(\omega)=R_{1p}(\omega)\,\omega/k_p(\omega)$$
$$R_{5p}(\omega)=(-1)^{p+1}k_p(\omega)/\Omega(\omega),\quad R_{6p}(\omega)=k_p(\omega)/\omega,\quad p=1,2$$

(an asterisk denotes the convolution of the functions with respect to τ). The value of m is chosen from the condition of absolute integrability of the integrand with respect to $\omega_1\in R^1$.

Note that, for the functions $G_j(|\xi|,\tau,n)$, the relations

$$\partial_\tau{}^pG_j(|\xi|,\tau,n)=G_j(|\xi|,\tau,n-p)$$
$$G_j(|\xi|,\tau,n)*f(\tau)=G_j(|\xi|,\tau,n+p)*\partial_\tau{}^p f(\tau) \tag{1.13}$$

are valid.

2. Let us find the properties of the functions $G_j(|\xi|,\tau,n)$ by which the solutions of the generalized Cauchy problem and its derivatives are expressed.

Using an asymptotic representation of the function $G_{jp}{}^L(|\xi|,\omega,n)$ as $\omega\to\infty$, the functions $G_{jp}(|\xi|,\tau,n)$ $(p=1,2)$ in the vicinity of the wavefronts $\tau=|\xi|/c_\pm$ can be expressed as

$$G_{j1}(|\xi|,\tau,n)=f_j{}^+\exp(-\eta_+|\xi|)\,D_{n+1}(\tau_+)\,E_+(\tau_+) \tag{2.1}$$
$$G_{j2}(|\xi|,\tau,n)=(-1)^j f_j{}^-\exp(-\eta_-|\xi|)\,D_{n+1}(\tau_-)\,E_-(\tau_-)$$
$$\tau_\pm=\tau-|\xi|/c_\pm,\quad \tau_\pm\to0,\ E_\pm(x)=[1+0(\eta_\pm c_\pm x)],\quad x\to0$$
$$\eta_\pm=\frac{\pm1}{2\eta c_\pm}\frac{c_0{}^2-c_\pm^2}{c_-{}^2-c_+{}^2},\quad f_1^\pm=\frac{c_+{}^2c_-{}^2}{c_-{}^2-c_+{}^2}$$
$$f_2^\pm=1,\quad f_3^\pm=c_\pm f_1^\pm,\quad f_5^\pm=f_1^\pm/c_\pm,\quad f_6^\pm=1/c_\pm$$

The function $D_\alpha(x)$ is a generalized function from the space $D_+'(0)$ which depends on the real parameter α, $-\infty<\alpha<+\infty$ /8/,

$$D_\alpha(x)=\begin{cases}H(x)x^{\alpha-1}/\Gamma(\alpha),\ \alpha>0\\ d^ND_{\alpha+N}/dx,\ \alpha\leqslant0,\ \alpha+N>0,\ N\ \text{is an integer.}\end{cases}$$

$H(x)=0$ for $x<0$, $H(x)=1$ for $x>0$, and $D_0(x)=\delta(x)$. In relations (2.1), we note the form of the attenuation coefficients $\eta_\pm$ as the velocity functions, from which follows the fact that a "wave hierarchy" is necessary for $\eta_\pm$ to be positive, that is condition (1.9) must be satisfied.

As $\tau\to\infty$, we can express the function $G_j(|\xi|,\tau,\eta)$ as

$$G_j(|\xi|,\tau,n)\sim\sum_{m=0}^{6}q_{jm}\left[\Gamma\left(n+\frac{3-m}{2}\right)\tau^{-n-(1-m)/2}\right]^{-1},\quad \tau\to\infty \tag{2.2}$$

where $[\Gamma(n+(3-m)/2)]^{-1}=0$ if $n+(3-m)/2=0,\ -1,\ -2,\ldots$; $\Gamma(x)$ is the gamma function. The non-zero coefficients q_{jm} have the form

$$q_{14}=-(-a_{21})^{-1/2}|\xi|,\quad q_{12}=2,\quad q_{22}=-(-a_{21})^{1/2}|\xi|$$
$$q_{33}=-(-a_{21}a_{03})^{-1/2},\quad q_{34}=(-a_{21})^{-3/2}$$
$$q_{36}=(-a_{21})^{-1/2}|\xi|^2/2$$
$$q_{52}=(-a_{21})^{-1/2},\quad q_{53}=-[(-a_{03}a_{21})^{1/2}/a_{21}+|\xi|]$$
$$q_{60}=(-a_{21})^{1/2},\quad q_{61}=a_{21}[(-a_{03}/a_{21})^{1/2}/a_{21}+|\xi|]$$

Notice that the function $G_j(|\xi|,\tau,n)\in D_+'(0)$ although its components $G_{jp}(|\xi|,\tau,n)\in D_+'(a_0)$, $p=1,2$, where $a_0=\max(0,-\omega_2^c)$.

The behaviour of the functions $G_j(|\xi|,\tau,n)$ at arbitrary distances behind the wave front, for large ξ and τ can be determined by the method of steepest descents (see /9/). Thus, in the vicinity of $\tau=|\xi|/c_0$ one can obtain the asymptotic formula

$$G_j(|\xi|,\tau,n)\sim q_jE(\tau_0)*F_j(\tau,n),\quad \tau_0\to0,\quad \tau c_0/\gamma\to\infty \tag{2.3}$$
$$E(\tau_0)=\exp[-(\tau_0/\tau_1)^2][\pi^{1/2}\tau_1]^{-1},\quad \tau_0=\tau-|\xi|/c_0$$
$$\tau_1=(4|\xi|\gamma/c_0{}^3)^{1/2}$$
$$F_j(\tau,n)=D_n(\tau),\quad j=1,3,5;\quad F_j(\tau,n)=D_{n+1}(\tau)$$
$$j=2,6$$
$$q_1=-\eta c_+{}^2c_-{}^2/c_0{}^2,\quad q_2=1,\quad q_3=-\eta c_+{}^2c_-{}^2/c_0$$
$$q_5=-\eta c_+{}^2c_-{}^2/c_0{}^3,\quad q_6=1/c_0,\quad \gamma=\eta(c_0{}^2-c_+{}^2)(c_-{}^2-c_0{}^2)/(2c_0{}^2)$$

The functions $G_j(|\xi|, \tau, n)\ (j = 1, 2, 3, 5, 6)$ can be expressed in the form of integrals over the segments which connect the branch points $\omega = \omega_\pm,\ \omega = 0,\ \omega = -i/\eta$ of the function $k_1(\omega)$ and $k_2(\omega)$.

We shall give a representation of the functions $G_1(|\xi|, \tau, 0)$, $G_1(|\xi|, \tau, 1)$, $G_2(|\xi|, \tau, 0)$, which we will use in discussing specific problems,

$$
\begin{aligned}
&G_j(|\xi|, \tau, n) = I_j(|\xi|, \tau, n)\,[H(\tau_-) - H(\tau_+)] + \\
&\quad I_{j1}(|\xi|, \tau, n)\,H(\tau_+) + (-1)^j I_{j2}(|\xi|, \tau, n)\,H(\tau_-) \\
&(j = 1, 2,\ n = 0, 1) \\
&I_j(|\xi|, \tau, n) = \frac{1}{\pi}\exp(-\omega_2^\circ\tau)\int_{\delta_2}^{\omega_1^\circ}[\Psi_{jn}^+\exp(-\alpha_+|\xi|) + \\
&\quad \Psi_{jn}^-\exp(-\alpha_-|\xi|)]\,dx \\
&\Psi_{10}^\pm = -l_0^{-1}\cos\varphi_\pm,\quad \Psi_{11}^\pm = (\omega_2^\circ\cos\varphi_\pm + x\sin\varphi_\pm)\,l_0^{-1}k_0^{-2} \\
&\Psi_{20}^\pm = \pm l_0\Psi_{11}^\pm,\quad l_0 = [(\omega_1^\circ)^2 - x^2]^{1/2},\quad k_0 = [(\omega_2^\circ)^2 + x^2]^{1/2} \\
&\varphi_\pm = |\xi|\,\gamma_\pm - \tau x,\quad \gamma_\pm = 2^{-1/2}[(Z^\circ + X_\pm^\circ)^{1/2}K_+ + \\
&\quad (Z^\circ - X_\pm^\circ)^{1/2}K_-] \\
&K_\pm = (k_0 \pm \omega_2^\circ)^{1/2},\quad \alpha_\pm = 2^{-1/2}[(Z^\circ + X_\pm^\circ)^{1/2}K_- - \\
&\quad (Z^\circ - X_\pm^\circ)^{1/2}K_+] \\
&Z^\circ = [(X_\pm^\circ)^2 + (Y^\circ)^2]^{1/2},\quad X_\pm^\circ = \omega_2^\circ a_{22} - a_{21} \pm p_1^{1/2}l_0 \\
&Y^\circ = -xa_{22} \\
&I_{jm}(|\xi|, \tau, n) = \frac{1}{\pi}\Big[(-1)^j\delta_{m2}\int_{\delta_1}^{1/\eta}\frac{\exp(-\tau x)}{(-x)^{n+1}}\theta_{jn}\sin(|\xi|(Ax)^{1/2})\,dx + \\
&\quad \delta_{m1}\int_{\delta_0}^{\delta_1}\frac{\exp(-\tau x)}{(-x)^{n+1}}\theta_{jn}\sin(|\xi|(Ax)^{1/2})\,dx - (-1)^j d_{jn},\quad m = 1, 2 \\
&k^\circ = [(\omega_2^\circ - x)^2 + (\omega_1^\circ)^2]^{1/2},\quad \theta_{11} = \theta_{10} = -x/k^\circ,\quad \theta_{20} = 1 \\
&A = (k^\circ p_1^{1/2} - a_{21} + xa_{22})/2,\quad \delta_{mj} = 1,\quad j = m,\quad \delta_{mj} = 0 \\
&j \neq m \\
&d_{10} = 0,\quad d_{20} = -1,\quad d_{11} = \begin{cases} -\operatorname{sign}(\omega_2^\circ)\,p_1^{1/2}/a_{21}, & |\omega_2^\circ| \geqslant \delta_0 \\ -2\pi p_1^{1/2}a_{21}^{-1}\arcsin(\omega_2^\circ/\delta_0), & |\omega_2^\circ| \leqslant \delta \end{cases} \\
&\delta_1 = \begin{cases} \delta_0, & \omega_2^\circ \leqslant \delta_0 \\ \omega_2^\circ, & \delta_0 \leqslant \omega_2^\circ \leqslant \eta^{-1} \\ \eta^{-1}, & \eta^{-1} \leqslant \omega_2^\circ \end{cases},\quad \delta_2 = \begin{cases} 0, & |\omega_2^\circ| \geqslant \delta_0 \\ [(\delta_0)^2 - (\omega_2^\circ)^2]^{1/2}, & |\omega_2^\circ| \leqslant \delta_0 \end{cases}
\end{aligned}
\tag{2.4}
$$

(δ_1 and δ_2 are alternative integration limits which depend on the coefficients of Eq. (1.1), and δ_0 is an arbitrarily small quantity).

Notice that this representation holds for arbitrary values of the coefficients of Eq. (1.1), which satisfy condition (1.6) and (1.7).

As $a_{04} \to 0$, Eq. (1.1) ceases to be hyperbolic but representations (2.2), (2.3) and (2.4) for the functions $G_j(|\xi|, \tau, n)$ remain in force, it being necessary to take into account that $c_- \to \infty$, $c_+ \to (-a_{22})^{-1/2}$, $\eta c_-^2 \to -a_{22}/a_{03}$ as $a_{04} \to 0$. For $a_{04} = 0$, we must replace the asymptotic expression (2.1) by the following:

$$
\begin{aligned}
&G_{j2}(|\xi|, \tau, n) \sim (-1)^j f_j^\circ D_n(\tau) * U_j(|\xi|, \tau),\quad \tau \to +0 \\
&U_1(|\xi|, \tau) = U_2(|\xi|, \tau) = \operatorname{erfc}(v),\quad U_3(|\xi|, \tau) = \\
&\quad (\pi\tau)^{-1/2}\exp(-v^2) \\
&U_5(|\xi|, \tau) = U_6(|\xi|, \tau) = 2\,[\tau^{1/2}U_3(|\xi|, \tau) - \\
&\quad vU_1(|\xi|, \tau)]\,\tau^{1/2} \\
&v = (-a_{03}/a_{22})^{1/2}(|\xi|/2)/\tau^{1/2},\quad f_1^\circ = -1/a_{22},\ f_6^\circ = \\
&\quad (-a_{03}/a_{22})^{1/2} \\
&f_2^\circ = 1,\quad f_3^\circ = f_1^\circ/f_6^\circ,\quad f_5^\circ = f_6^\circ f_1^\circ
\end{aligned}
\tag{2.5}
$$

3. Let us consider the dynamic processes in a thermoelastic half-space described by the following system of equations (see /2/):

$$
\begin{aligned}
&\{L_1u\} - \{L_3\theta\} = 0,\quad \{L_2\theta\} - \{L_4u\} = 0,\quad x > 0,\quad t > 0 \\
&L_1 = \partial_x^2 - \rho(\lambda + 2\mu)^{-1}\partial_t^2,\quad L_2 = \partial_x^2 - \varkappa^{-1}L_0\partial_t,\quad L_3 = \\
&\quad \beta(\lambda + 2\mu)^{-1}\partial_x \\
&L_4 = \eta_0 L_0\partial_{tx}^2,\quad L_0 = 1 + t_r\partial_t;\quad \beta = \alpha_\theta(3\lambda + 2\mu)
\end{aligned}
\tag{3.1}
$$

Here, $u(x, t)$ is the displacement, $\theta(x, t)$ is the temperature, α_θ is the coefficient of linear expansion, λ, μ are the Lamé coefficients, t_r is the relaxation time of the heat flux, η_0 is the coupling coefficient, $\varkappa$ is the thermal conductivity, ρ denotes the density, and $\partial_{tx}^2 = \partial^2/\partial t\partial x$. The derivatives in curly brackets should be understood in the classic sense, and the remaining ones in the generalized sense.

The surface of the half-space is subjected to a mechanical and a thermal action as follows:

$$\sigma(x, t) = (\lambda + 2\mu)\{\partial_x u\} - \beta\theta = P(t) \quad \text{for} \quad x = +0, \quad t > 0 \tag{3.2}$$
$$\theta(x, t) = T(t) \quad \text{for} \quad x = +0, \quad x > 0$$

where σ is the normal stress. We assume the initial condition to be zero:

$$u = \theta = \{\partial_t u\} = \{\partial_t\theta\} = 0 \quad \text{for} \quad t = +0, \quad x > 0 \tag{3.3}$$

Extending the functions $u(x, t), \theta(x, t)$ by zero with respect to t for $t < 0$, respectively evenly and oddly with respect to x for $x < 0$, and taking into account the connection between the classical derivatives and the generalized derivatives (see /8/), we reduce problem (3.1), (3.2), (3.3) when $x > 0$, to the equivalent generalized Cauchy problem for the system

$$L_1 u - L_3\theta = \psi_1, \quad L_2\theta - L_4 u = \psi_2, \quad x \in R^1, \quad t \in R^1 \tag{3.4}$$
$$\psi_1 = 2\delta(x) P(t)/(\lambda + 2\mu), \quad \psi_2 = 2\delta^{(1)}(x) T(t)$$

The solution of this system can be expressed by the functions φ_j $(j = 1, 2)$

$$u = L_2\varphi_1 + L_3\varphi_2, \quad \theta = L_4\varphi_1 + L_1\varphi_2$$

which are defined as solutions of the generalized Cauchy problem for the equations

$$(L_1L_2 - L_3L_4)\varphi_j = \psi_j \quad (j = 1, 2)$$

and whose properties in dimensionless variables are discussed above. Taking into account the relations

$$a_{21} = -(1 + \varepsilon), \quad a_{22} = -1 - M^2(1 + \varepsilon), \quad a_{03} = 1, \quad a_{04} = M^2$$
$$\eta = M^2, \quad \gamma = \varepsilon/(2c_0^2), \quad c_0 = (1 + \varepsilon)^{1/2}, \quad M = c_1/c_q$$
$$\varepsilon = \beta\eta_0\varkappa/(\lambda + 2\mu), \quad c_1 = (\lambda + 2\mu)^{1/2}\rho^{-1/2}, \quad c_q = (\varkappa/t_r)^{1/2}$$
$$M_0 = 1 - M^2(1 + \varepsilon), \quad P(t) = \sigma_* P_0(\tau), \quad T(t) = \theta_* T_0(\tau)$$
$$\sigma_* = \beta\theta_*, \quad \tau = c_1^2 t/\varkappa, \quad \xi = c_1 x/\varkappa$$

we can represent the dimensionless stress σ/σ_* and temperature θ/θ_* in the form

$$\sigma(\xi, \tau)/\sigma_* = [\sigma_\sigma(\xi, \tau) + \sigma_\theta(\xi, \tau)] \operatorname{sign} \xi \tag{3.5}$$
$$\theta(\xi, \tau)/\theta_* = [\theta_\sigma(\xi, \tau) + \theta_\theta(\xi, \tau)] \operatorname{sign} \xi$$

where the dimensionless stresses σ_σ and temperature θ_σ (the stress σ_θ and the temperature θ_θ) caused by mechanical action (thermal action), when the zero temperature (zero stress) is specified, in the form

$$\sigma_\sigma(\xi, \tau) = {}^1/_2\,[M_0 G_1(|\xi|, \tau, -1) - c_0^2 G_1(|\xi|, \tau, 0) + G_2(|\xi|, \tau, -1)] * P_0(\tau) \tag{3.6}$$
$$\theta_\sigma(\xi, \tau) = \varepsilon\,[G_1(|\xi|, \tau, 0) - M^2 G_1(|\xi|, \tau, -1)] * P_0(\tau)$$
$$\sigma_\theta(\xi, \tau) = G_1(|\xi|, \tau, -1) * T_0(\tau)$$
$$\theta_\theta(\xi, \tau) = {}^1/_2\,[-M_0 G_1(|\xi|, \tau, -1) - c_0^2 G_1(|\xi|, \tau, 0) + G_2(|\xi|, \tau, -1)] * T_0(\tau)$$

Let us analyse the behaviour of a single perturbation of the stress and temperature specified at the boundary $\xi = 0$ of the thermoelastic space $\xi > 0$.

The first (second) perturbation of the stresses σ_σ^- (σ_σ^+) and σ_θ^- (σ_θ^+) and temperatures θ_σ^- (θ_σ^+) and θ_θ^- (θ_θ^+) propagates with velocity c_- (c_+)

$$\sigma_\sigma^\mp(\xi, \tau) = {}^1/_2\,(1 \mp M_0 p_1^{-1/2}) P_0(\tau_\mp) \exp(-\eta_\mp\xi) \tag{3.7}$$
$$\sigma_\theta^\mp(\xi, \tau) = \mp p_1^{-1/2} T_0(\tau_\mp) \exp(-\eta_\mp\xi)$$
$$\theta_\sigma^\mp(\xi, \tau) = \mp\varepsilon M^2 p_1^{-1/2} P_0(\tau_\mp) \exp(-\eta_\mp\xi)$$
$$\theta_\theta^\mp(\xi, \tau) = {}^1/_2\,(1 \pm M_0 p_1^{-1/2}) T_0(\tau_\mp) \exp(-\eta_\mp\xi), \quad \tau_\mp \to 0$$

that is, for a fixed ξ the stress and temperature in the vicinity of the wavefront maintain the form of the input signal, but these perturbations decrease exponentially and become negligible at a distance of $1/\eta_\mp$.

If we assume that the time scale τ_σ is characteristic for the initial perturbation $P_0(\tau)$, and the scale τ_θ describes $dT_0(\tau)/d\tau$ that is $P_0(\tau), dT_0(\tau)/d\tau$ are isolated pulses with the characteristic durations τ_σ and τ_θ, then at distances $\tau_1 \gg \tau_\sigma$, $\tau_1 \gg \tau_\theta$, $\tau_1^2 = 2\xi\varepsilon c_0^{-6}$ the behaviour of the stress and temperature perturbations is described by the asymptotic expressions

$$\sigma_\sigma \sim E(\tau_0) A_\sigma, \quad \sigma_\theta \sim -c_0^{-2} E(\tau_0) A_\theta$$
$$\theta_\sigma \sim -\varepsilon c_0^{-2} E(\tau_0) A_\sigma, \quad \theta_\theta \sim \varepsilon c_0^{-4} E(\tau_0) A_\theta, \quad \tau/\varepsilon \to \infty, \quad \tau \sim \xi/c_0$$
$$A_\sigma = \int_0^\infty P_0(\tau)\,d\tau < \infty, \quad A_\theta = \int_0^\infty \frac{d}{d\tau} T_0(\tau)\,d\tau < \infty$$

Thus, in the course of time the majority of the perturbations become bell-shaped and begin to propagate with the adiabatic velocity c_0 ($c_s = c_1 c_0$ is the adiabatic velocity of propagation of an expansion wave). These waves spread, their width increases $\sim\tau_1$ and the amplitude drops $\sim\tau_1^{-1}$. This effect is a consequence of the singularities of the pulse propagation in the media with dispersion and absorption, and is common for all types of waves.

A numerical analysis was carried out for the case of the sudden application of a mechanical action ($P_0(\tau) = H(\tau)$) and a thermal shock /14/ ($T_0(\tau) = H(\tau)$), and for a large coupling coefficient $\varepsilon = 0.46$ (see /3, 15/), i.e. for the least favourable cases for numerical methods which were used in /16–19/ for solving similar problems, and for the asymptotic methods of expanding the solution in terms of small parameter. In such a case, in accordance with (3.6) we have

$$\begin{aligned}
\sigma_\sigma(\xi,\tau) &= {}^1/_2\,[M_0 G_1(\xi,\tau,0) - c_0^2 G_1(\xi,\tau,1) + G_2(\xi,\tau,0)] \\
\theta_\sigma(\xi,\tau) &= \varepsilon\,[G_1(\xi,\tau,1) + M^2 G_1(\xi,\tau,0)] \\
\sigma_\theta(\xi,\tau) &= G_1(\xi,\tau,0) \\
\theta_\theta(\xi,\tau) &= {}^1/_2\,[-M_0 G_1(\xi,\tau,0) + c_0^2 G_1(\xi,\tau,1) + G_2(\xi,\tau,0)]
\end{aligned} \tag{3.8}$$

where $G_j(\xi,\tau,n)$ are calculated from (2.4).

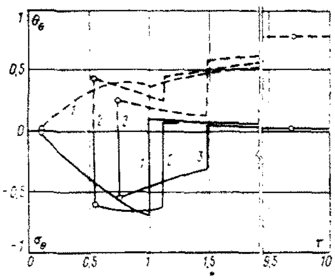

Fig.1

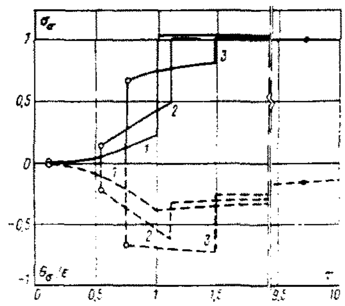

Fig.2

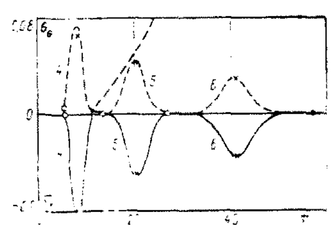

Fig.3

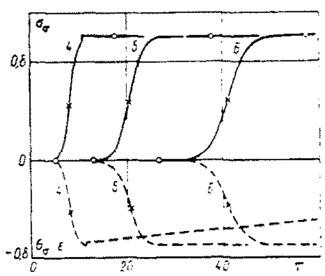

Fig.4

Figs.1 and 2 show, following (3.8) and (2.4), the variation of the dynamic stresses (solid lines) and temperatures (dashed lines) at a fixed cross-section $\xi = 1$, as a function of the dimensionless time τ for values of the parameter M equal to 0.1, 0.6 and 1.1 (curves 1–3). Figs.3 and 4 show the variation of the same quantities, as a function of the dimensionless time τ for $M = 0.6$ and $\xi = 10, 25, 50$ (curves 4–6). The circles, points and crosses show the resulting obtained by the asymptotic formulae (3.7), (2.2) and (2.3) respectively.

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REDUCTION OF THREE-DIMENSIONAL DYNAMICAL ELASTICITY THEORY PROBLEMS WITH ARBITRARILY LOCATED PLANE SLITS TO INTEGRAL EQUATIONS*

V.V. MYKHAS'KIV and M.V. KHAI

By generalizing a method described earlier /1/ for reducing three-dimensional dynamical problems of elasticity theory for a body with a slit to integral equations, integral equations are obtained for an infinite body with arbitrarily located plane slits. The interaction of disc-shaped slits located in one plane is investigated when normal external forces that vary sinusoidally with time (steady vibrations) are given on their surfaces.

Problems of the reduction of dynamical three-dimensional elasticity theory problems to integral equations for an infinite body weakened by a plane slit were examined in /1, 2/. The solution of the initial problem is obtained in /1/ by applying a Laplace integral transform in time to the appropriate equations and constructing the solution in the form of Helmholtz potentials with densities characterizing the opening of the slit during deformation of the body. The problem under consideration is solved in /2/ by using the fundamental Stokes solution /3/ with subsequent construction of the solution in the form of an analogue of the elastic potential of a double layer.

1. We consider an ealstic infinite body weakened by plane arbitrarily located slits whose opposite surfaces S_n^+ and S_n^- $(n = 1, 2, \ldots, N)$ are subjected to selfequilibrated external forces varying with time t. We consider the initial conditions of the problem to be zero.

We select a basic Cartesian coordinate system $Ox_1x_2x_3$ with origin O at an arbitrary point of the body and local coordinate systems $O_nx_{1n}x_{2n}x_{3n}$ $(n = 1, 2, \ldots, N)$ in such a way that the domain S_n which the n-th slit occupies would be contained in the coordinate plane $x_{1n}O_nx_{2n}$, while the values $x_{3n} = \pm 0$ (Fig.1) would correspond to the surfaces $S_n^\pm$. Let x denote the point with coordinates (x_1, x_2, x_3).

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